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Some Sahlqvist completeness results for Coalgebraic Logics

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Abstract. This paper presents a first step towards completeness-via-canonicity results for coalgebraic modal logics. Specifically, we consider the relationship between classes of coalgebras for ω-accessible endofunctors and logics defined by Sahlqvist-like frame conditions. Our strategy is based on conjoining two well-known approaches: we represent accessible functors as (equational) quotients of polynomial functors and then use canonicity results for boolean algebras with operators to transport completeness to the coalgebraic setting.

Keywords: Modal logic, coalgebraic modal logic, canonicity, completeness, Sahlqvist formula

1 Introduction

Coalgebras have gained popularity as an elegant and general framework to study and represent a wide variety of dynamical systems in computer science (see [12]) and even in physics (see [1]). In parallel to this area of research, the field of coalgebraic logic has emerged as a unifying framework for the many types of (modal) logics used to reason about dynamical systems (see [8] for an overview). One of the great insights into the relationship between coalgebras and coalgebraic logics, is that the class of all T-coalgebras for a functor T can always be characterised logically by its one-step behaviour, i.e. axioms and rules with nesting depth of modal operators uniformly equal to 1 (see [13]). However, once the transition type (i.e. the functor T) has been described logically in such a way, one may be interested in subclasses of T-coalgebras which are characterised by more complex axioms (such as transitivity for example) which we will refer to as *frame conditions*. The problem of logically characterising subclasses of the class of all T-coalgebras for an arbitrary functor T is by and large still open ([11] offers a solution for some of the standard frame conditions of classical modal logic).

This paper aims to isolate a large class of frame conditions which can be used to logically characterise proper subclasses of coalgebras, i.e. axioms giving a sound and complete description of certain classes of coalgebras. Our strategy is based on the following observations. Firstly, it is well known that accessible **Set** functors can be represented as (equational) quotients of polynomial functors. Secondly, the coalgebraic logics for polynomial **Set** functors turns out to be very closely related to Boolean Algebras with Operators (BAOs). Thirdly, there is a well developed theory of Sahlqvist formulae for general BAOs (see [6, 5, 14]). The first step will therefore be to show how Sahlqvist formulae can be imported into the coalgebraic logics of polynomial functors, the second step will be to show how they can then be transported to logics of general functors via the presentation.

The paper in organized as follows: in *Section 2* we will present the basic facts about BAOs and coalgebraic logics that are needed for the rest of the paper. This will be very succinct and the reader is referred to [7, 6, 5, 14, 9, 8] for further details. The ∇-style of coalgebraic logic requires some notational discipline, and the notation of [9] is presented in detail. The section concludes with our first Sahlqvist-like completeness result for polynomial functors. *Section 3* will first address the idea of presenting a functor T with a polynomial functor S (again we will present the bare minimum and the reader is referred to [2] for all the details), and then explore what this means for coalgebraic logics. In *Section 4* we present the main technical result of the paper, the Translation Theorem, which relates the derivability in the logic associated to a functor T to that in the logic of the functor S presenting it. Finally, in *Section 5* we gather all our results together and present a Sahlqvist completeness theorem for coalgebraic logics.

2 BAOs and Coalgebraic Logics

We start with some notation, basic definitions and facts about BAOs and coalgebraic modal logics. Readers familiar with this material can safely move to Example 1 which should offer a first glimpse at what this paper aims to achieve.

Boolean Algebras with Operators (BAOs). We roughly follow the terminology of [6] which itself is based on the seminal paper [7]. A **Boolean Algebra with Operator (BAO)** is a Boolean Algebra (BA) $\mathfrak{A}$ together with functions $f_\sigma : A^{\mathrm{ar}(\sigma)} \to A$ where A is the set underlying $\mathfrak{A}$ and σ is an element of a signature (Σ, ar) with arity map $\mathrm{ar} : \Sigma \to \mathbb{N}$. The maps f_σ are required to preserve joins in each of their arguments, in which case they are known as **operators**. The BAOs with a given signature Σ, together with the BA-morphism preserving operators in the obvious way, form a category which we will call $\mathbf{BAO}(\Sigma)$. As shown in [7], every BA $\mathfrak{A}$ can be embedded in a unique Complete Atomic Boolean Algebra (CABA) $\mathfrak{A}^\varepsilon$ called its **canonical extension** and which has the property that (1) every atom of $\mathfrak{A}^\varepsilon$ is a meet of elements of $\mathfrak{A}$ and (2) every subset in A (the set underlying $\mathfrak{A}$) whose join in $\mathfrak{A}^\varepsilon$ is $\top$, has a *finite* subset whose join in $\mathfrak{A}$ is also $\top$. This result can be extended to include operators (in fact any monotone map), viz. any BAO $\mathcal{A}$ can be embedded in a BAO $\mathcal{A}^\varepsilon$ - its canonical extension - whose underlying BA is the canonical extension of that of $\mathcal{A}$. This result is of fundamental importance because the category of CABAs is dual to the category **Set** in which models live.

Sahlqvist formulae in a BAO. Let us fix a BAO $\mathcal{A} = (\mathfrak{A}, \{f_\sigma \mid \sigma \in \Sigma\})$. We define a **$\Sigma$-term** to be an element of the algebra freely generated by the elements of $\mathfrak{A}$

and the operators $f_\sigma, \sigma \in \Sigma$. Following [6] and [5], we define a **Sahlqvist term** to be a Σ-term of the form:

$$u[v_1, \ldots, v_n, \neg w_1, \ldots, \neg w_m] \tag{1}$$

where (i) u is a *strictly positive* $m+n$-ary term, i.e. contains no negations, (ii) the v_i's are terms of the shape $v_i = \sigma_1^d(\ldots(\sigma_k^d(x))\ldots)$ where each $\sigma_i^d = \neg\sigma_i\neg$ is the dual of a unary operator $\sigma_i \in \Sigma$, and (iii) the w_k's ($1 \leq k \leq m$) are *positive* terms, i.e. all variables in w_k must occur in the scope of an even number of complementation symbols. A **Sahlqvist equation** is an equation of the type $s = 0$ where s is a Sahlqvist term. A **Sahlqvist inequality** (or Sahlqvist formula in the context of algebras of terms) is an inequality of the type $s \leq t$ where t is positive. As shown in [6], all Sahlqvist identities are canonical, i.e. if $s = 0$ holds in $\mathcal{A}$, then it holds in its canonical extension $\mathcal{A}^\varepsilon$.

Coalgebraic Logics. Coalgebraic logics come in two flavours which we now introduce very succinctly. In both cases V denotes a set of propositional variables. We start with the **predicate lifting style** of coalgebraic logic. A coalgebraic language L_T has a syntax given by

$$a ::= p \mid \bot \mid \neg a \mid a \wedge b \mid \sigma(a_1, \ldots, a_n)$$

where $p \in V$ and $\sigma \in \Sigma$ are modal operators belonging to a signature (Σ, ar). Note that we're using the notational convention of [9] where the lower case Roman letters a, b, c stand for formulae. Such a language is interpreted in terms of coalgebras and predicate liftings. Given a standard **Set**-endofunctor T (we will assume throughout the paper that all functors are standard), a **coalgebra** is a pair (W, γ) where W is a set (of worlds) and $\gamma : W \to TW$ is a transition map (T defines the 'transition type'). Each modal operator σ is interpreted by a **predicate lifting**, i.e. a natural transformation $[\![\sigma]\!] : \mathcal{Q}^n \to \mathcal{Q}T$ where $\mathcal{Q}$ is the contravariant powerset functor. Intuitively, predicate liftings 'lift' n-tuples of predicates (i.e. subsets, hence the powerset functor) to a predicate on transitions (hence $\mathcal{Q}T$). A **coalgebraic model** - or T-model - is a triple $\mathcal{M} = (W, \gamma, \pi)$ where $\pi : W \to \mathcal{P}(V)$ is a valuation. The notion of **truth** of a formula a at a point $w \in W$ is defined inductively in the usual manner for propositional variables and boolean operators, and by

$$\mathcal{M}, w \models \sigma(a_1, \ldots, a_n) \text{ iff } \gamma(w) \in [\![\sigma]\!]_W([\![a_1]\!], \ldots, [\![a_n]\!])$$

for modal operators, where $[\![a_i]\!]$ is the interpretation of a_i in W. A formula a is **satisfiable** in $\mathcal{M}$ if there exists $w \in W$ such that $\mathcal{M}, w \models a$. A **coalgebraic frame** - or T-frame - is just a T-coalgebra (W, γ) and a formula a is **valid** on the frame if for any valuation π, a is true at every point in the model (W, γ, π). The **∇-style** of coalgebraic logic (also known as Moss style, or coalgebraic logic for the cover modality) has a very different flavour. We recall the basic definitions and results, and refer to [9] for a very good and very thorough overview of the topic. Since the language involves objects of many types, our notation follows the conventions of [9] very closely to avoid confusion. We start by fixing a weak-pullback preserving functor T and we define $T_\omega = \bigcup\{TY \mid Y \subseteq X \text{ finite}\}$, the finitary version of T. The coalgebraic language $\mathcal{L}_T$ induced by T is given by:

$$a ::= p \mid \neg a \mid \bigwedge \phi \mid \bigvee \phi \mid \nabla\alpha$$

where $p \in V$, $\phi \in \mathcal{P}_\omega \mathcal{L}_T$ and $\alpha \in T_\omega \mathcal{L}_T$. $\bigvee\emptyset$ defines $\bot$. For any $\alpha \in T_\omega \mathcal{L}_T$ we define the **base** of α by $\mathsf{Base}^T(\alpha) = \bigcap\{U \subseteq \mathcal{L}_T \mid \alpha \in TU\}$, i.e. the set of immediate subformulae of $\nabla\alpha$. Given a T-model $\mathcal{M} = (W, \gamma, \pi)$, the truth relation $\models \subseteq W \times \mathcal{L}_T$ is inductively defined for any world $w \in W$ and formula $a \in \mathcal{L}_T$ by the usual clauses for atomic propositions and propositional connectives and

$$\mathcal{M}, w \models \nabla\alpha \text{ iff } \gamma(w)\bar{T}(\models)\alpha$$

where $\bar{T}(\models) \subseteq TW \times T\mathcal{L}_T$ is the *relation lifting* of the truth-relation $\models \subseteq W \times \mathcal{L}_T$ (see [9] for an extensive discussion of relation liftings).
Coalgebraic Logic is weakly complete (see [9]) with respect to the 2-dimensional Hilbert system which we call $\mathsf{KKV}(T)$ and is given by the axioms and rules:

$$\frac{}{a \le a} \qquad (\text{Cut})\ \frac{a \le c \qquad c \le b}{a \le b}$$

$$(\textstyle\bigvee L)\ \frac{\{a \le b \mid a \in \phi\}}{\bigvee \phi \le b} \qquad (\textstyle\bigvee R)\ \frac{a \le b}{a \le \bigvee \phi}\ b \in \phi$$

$$(\textstyle\bigwedge L)\ \frac{a \le b}{\bigwedge \phi \le b}\ a \in \phi \qquad (\textstyle\bigwedge R)\ \frac{\{a \le b \mid b \in \phi\}}{a \le \bigwedge \phi}$$

$$(\neg E)\ \frac{\bigwedge\{\phi \cup \{\neg a\}\} \le \bigvee \psi}{\bigwedge \phi \le \bigvee\{\psi \cup \{a\}\}} \qquad (\neg I)\ \frac{\bigwedge\{\phi \cup \{a\}\} \le \bigvee \psi}{\bigwedge \phi \le \bigvee\{\psi \cup \{\neg a\}\}}$$

$$(\text{Distributivity})\ \frac{}{\bigwedge\{\bigvee \phi \mid \phi \in X\} \le \bigvee\{\bigwedge \mathsf{rng}(\gamma) \mid \gamma \in \mathsf{Choice}(X)\}}$$

$$(\nabla 1)\ \frac{\{a \le b \mid (a,b) \in R\}}{\nabla\alpha \le \nabla\beta}\ (\alpha, \beta) \in \bar{T}R$$

$$(\nabla 2)\ \frac{\{\nabla(T\bigwedge)(\Phi) \le b \mid \Phi \in SRD(A)\}}{\bigwedge\{\nabla\alpha \mid \alpha \in A\} \le b} \qquad (\nabla 3)\ \frac{\{\nabla\alpha \le b \mid \alpha \bar{T}{\in} \Phi\}}{\nabla(T\bigvee)(\Phi) \le b}$$

where $a, b \in \mathcal{L}_T$, $\phi, \psi \in \mathcal{P}_\omega \mathcal{L}_T$, $X \in \mathcal{P}_\omega \mathcal{P}_\omega \mathcal{L}_T$, $\alpha, \beta \in T_\omega \mathcal{L}_T$, $\Phi \in T_\omega \mathcal{P}_\omega \mathcal{L}_T$, $A \in \mathcal{P}_\omega T_\omega \mathcal{L}_T$. The set $\mathsf{Choice}(X)$ is the set of choice functions on X, i.e. the maps $\gamma : X \to \mathcal{L}_T$ such that $\gamma(\phi) \in \phi$, and rng denotes the range of the function. $R \subseteq \mathcal{L}_T \times \mathcal{L}_T$ is any relation and $\bar{T}R$ is its lifting. Finally $SRD(A)$ is the set of so-called 'slim redistributions' of A. This last concept is important, and we therefore define it in extenso. A **redistribution** of $A \in \mathcal{P}_\omega T_\omega \mathcal{L}_T$ is an element Φ

of $T_\omega \mathcal{P}_\omega \mathcal{L}_T$ which 'contains' all the elements of A as lifted members, i.e. $\alpha \,\overline{T}\!\in \Phi$ for all $\alpha \in A$. It is called **slim** if it is build from the direct subformulae of the elements of A, i.e. if $\Phi \in T_\omega \mathcal{P}_\omega(\bigcup_{\alpha \in A} \mathsf{Base}(\alpha))$.

To help the reader digest this rather heavy load of definitions, let us look at an example which will cover both BAOs and coalgebraic logics.

Example 1. Given a signature (Σ, ar), we define a functor $S_\Sigma : \mathbf{Set} \to \mathbf{Set}$ by

$$S_\Sigma X = \coprod_{n \in \omega} \Sigma_n \times X^n$$

where Σ_n is the set regrouping all operation symbols $\sigma \in \Sigma$ of arity n. Any functor of this shape is called a **polynomial functor**. We write $U : \mathbf{BA} \to \mathbf{Set}$ for the forgetful functor, and $F : \mathbf{Set} \to \mathbf{BA}$ for its left adjoint (the associated free construction). This allows us to lift set-functors $T : \mathbf{Set} \to \mathbf{Set}$ to the category of boolean algebras by putting $\mathsf{T} = FTU : \mathbf{BA} \to \mathbf{BA}$. In particular, every signature Σ induces the functor $\mathsf{S}_\Sigma : \mathbf{BA} \to \mathbf{BA}$ defined by

$$\mathsf{S}_\Sigma \mathfrak{A} = F S_\Sigma U \mathfrak{A} = F\left(\coprod_{\sigma \in \Sigma} A^{\mathrm{ar}(\sigma)}\right)$$

where $A = U\mathfrak{A}$. From now on we will drop the Σ subscript if there is no risk of confusion. It is easy to see by the freeness of the construction of S that the category $\mathbf{Alg}(\mathsf{S})$ of S-algebras in $\mathbf{BA}$ is isomorphic to the category of boolean algebras $\mathfrak{A}$ with maps $f_\sigma : A^n \to A$ where n is the arity of σ. But this is almost the category $\mathbf{BAO}(\Sigma)$ defined above! The only difference is that the maps do not have to be operators, but we will return to this in an instant.
If we now turn our attention to coalgebraic logic, we can use the signature Σ to define a coalgebraic language L_S in the predicate lifting style defined above. It is relatively straightforward to see that the Lindenbaum-Tarski algebra of L_S, which we will denote $\mathcal{A}_S$, is the initial object in $\mathbf{Alg}(\mathsf{S})$, or equivalently, the free BAO of Σ-terms as defined above but with the preservation of joins not being enforced. We would like to interpret L_S in S-models by reading $\mathcal{M}, w \models \sigma(a_1, \ldots, a_n)$ as 'w has a σ-tuple successor and a_i holds at the i^{th} component of this successor'. The predicate liftings are then defined by:

$$[\![\sigma]\!]_W : (\mathcal{P}W)^n \to \mathcal{P}SW, (U_1, \ldots, U_n) \mapsto \{\sigma(x_1, \ldots, x_n) \mid x_i \in U_i, 1 \le i \le n\}$$

The obvious question now is: what axioms will give a sound and complete axiomatization of $\mathbf{CoAlg}(S)$, the class of all S-coalgebras? It is not too difficult to find this list of axioms from scratch and then use a canonical model construction to prove completeness, however, the fact that we're using Σ in our syntax and our semantics suggests turning our attention to the ∇-flavour of coalgebraic logic.
There is an obvious one-to-one correspondence between the language L_S defined above and the ∇-style language $\mathcal{L}_S$ induced by S. Since $S = S_\omega$ and since $\sigma(a_1 \ldots, a_n)$ can be seen as an element of $S\mathsf{L}_S$, we can recursively add ∇ in

front of every modal operator σ in order to get a ∇-style formula. Conversely, by recursively removing every ∇ from a formula in $\mathcal{L}_S$ we get a formula in L_S (see [10] for a detailed discussion of translations between the two flavours of coalgebraic logic). We now have rules that provide us with a sound and (weakly) complete axiomatization of $\mathbf{CoAlg}(S)$, namely:

$$(\nabla 1)_S \quad \frac{\{a_i \leq b_i \mid 1 \leq i \leq n\}}{\nabla\sigma(a_1, \ldots, a_n) \leq \nabla\sigma(b_1, \ldots, b_n)}$$
$$(\nabla 2)_S \quad \bigwedge\{\nabla\sigma(a_1, \ldots, a_n) \mid \sigma(a_1, \ldots, a_n) \in A\} = \nabla\sigma(\bigwedge \pi_1[A], \ldots, \bigwedge \pi_n[A])$$
$$(\nabla 3)_S \quad \nabla\sigma(\bigvee \phi_1, \ldots, \bigvee \phi_n) = \bigvee\{\nabla\sigma(a_1, \ldots, a_n) \mid a_i \in \phi_i\}$$

where $=$ means both $\leq$ and $\geq$, $a_i \in \mathcal{L}_S, A \in \mathcal{P}_\omega S\mathcal{L}_S, \phi_i \in \mathcal{P}_\omega \mathcal{L}_S$. Let us discuss these rules and axioms briefly. The $(\nabla 1)_S$ axiom takes this simplified form because relation lifting by polynomial functors is very simple: if $R \subseteq X \times X$, then $(\alpha, \beta) \in \bar{S}R$ only if α, β lie in the same part of the co-product SX and each component of the tuple α is R-related to the corresponding component of β. For $(\nabla 2)_S$, it is easy to see from the definition of slim redistribution that $SRD(A)$ is empty if A contains elements lying in different parts of the co-products, hence the presence of σ-terms only. Moreover, it is not too hard to check that if $\Phi, \Psi \in SRD(A)$ and $(\Psi, \Phi) \in \bar{S} \subseteq$, then $\nabla S \bigwedge \Phi \leq \nabla S \bigwedge \Psi$, and since $\sigma(\pi_1[A], \ldots, \pi_n[A])$ is a lifted subset of all other elements of $SRD(A)$, $\nabla\sigma(\bigwedge \pi_1[A], \ldots, \bigwedge \pi_n[A]) \leq b$ implies $\nabla S \bigwedge \Phi \leq b$ for all other $\Phi \in SRD(A)$. Finally, and most importantly for our purpose, $(\nabla 3)_S$ is just another way of saying that ∇ preserves joins in each of its arguments. Of course the number of arguments of ∇ can vary, but by trivially translating into L_S we get that σ - which has a fixed number of arguments - preserves joins in each of its arguments. To see this for the first argument for example, just take $\phi_1 = \{a_1, b_1\}$ and $\phi_i = \{a_i\}$ for $1 < i \leq n$, and note that the premise of the $(\nabla 3)$ rule is a finite set for which we can take the join[1].

Let us denote by K_S the predicate-lifting style logic defined by the trivial translations of the axioms $(\nabla 1)_S - (\nabla 3)_S$ from $\mathcal{L}_S$ to L_S and any axiomatization of propositional logic. It is easy to see that the semantics of $\mathcal{L}_S$ is essentially the same as the semantics we defined for L_S and since we know that $\mathsf{KKV}(S)$ is sound and complete w.r.t. the class of all S-coalgebras and that $\mathsf{KKV}(S)$ and K_S are in bijective correspondence, we can conclude that K_S is also sound and weakly complete w.r.t. the class of all S-coalgebras. The conclusion of this example is therefore that if we look at the *logic* K_S, rather than at the *language* L_S, then the Lindenbaum-Tarski algebra $\mathcal{A}_S$ of K_S is a bona fide BAO since preservation of joins has been enforced by the $(\nabla 3)_S$ axiom.

The previous example suggests our first Sahlqvist completeness theorem. But let us first define a notion of Sahlqvist formula for the coalgebraic logic of a polynomial functor.

[1] We refer the reader to [9] to see that the converse of $(\nabla 2)$ and $(\nabla 3)$ (i.e. with the inequalities going in the opposite direction) are derivable using $(\nabla 1)$ and the fact that S preserves weak-pullbacks.

Definition 2. Let a be a formula in $\mathsf{KKV}(S)$ for a polynomial functor S, and let a_S be its trivial translation into the predicate-lifting-style logic K_S. Then a (and a_S) will be called a **polynomial Sahlqvist formula** if (the equivalence class of) a_S in the Lindenbaum-Tarski algebra $\mathcal{A}_S$ of K_S is a Sahlqvist formula as defined above (following [6]).

Our first Sahlqvist completeness theorem is shown by following the well-trodden path of completeness-via-canonicity proofs (for more details on this technique we refer the reader to Chapter 4 and 5 of the classic [4]). Assume that Σ is a signature defining a polynomial functor S and that C is a set of frame conditions, then we want to endow the set of ultrafilters of the Lindenbaum-Tarski algebra $\mathcal{A}_S(C)$ of $\mathsf{K}_S + C$ (i.e. the BAO of formulae L_S quotiented by equivalence under $\mathsf{K}_S + C$) with the structure of an S-coalgebra. The natural transition map to use is $\gamma_c : \mathsf{Uf}(\mathcal{A}_S(C)) \to S\mathsf{Uf}(\mathcal{A}_S(C))$ defined by

$$\gamma_c(\phi) = \sigma(\psi_1, \ldots, \psi_n) \tag{2}$$

where $a_i \in \psi_i, 1 \leq i \leq n$ if $\sigma(a_1, \ldots, a_n) \in \phi$. However, when Σ is infinite some care must be taken due to the following fact. Consider the set

$$\zeta = \{\neg\sigma(\top, \ldots, \top) \mid \sigma \in \Sigma\}$$

The set ζ is K_S-consistent but γ_c is undefined on any ultrafilter containing it. In particular, any set of frame conditions containing ζ will lead to a situation where γ_c cannot be defined anywhere. The set ζ characterises precisely the set of ultrafilters for which γ_c is well-defined. Note that if Σ is finite, this problem does not arise, moreover ζ cannot be a subset of any finite set of frame conditions. But as we shall see later, infinite signatures and infinite sets of frame conditions will be very useful. This justifies the following definition.

Definition 3. We recursively define the collection $\mathcal{Z}$ of **deadlocking sets of formulae** as follows: $\zeta \in \mathcal{Z}$ and if $\zeta' \in \mathcal{Z}$, then for any operator $\sigma \in \Sigma, \mathrm{ar}(\sigma) = n$ and map $\chi : n \to \{1, 2\}$ s.th. $1 \in \mathsf{rng}(\chi)$

$$\{\sigma(\pi_{\chi(0)}(z, \top), \ldots, \pi_{\chi(n)}(z, \top) \mid z \in \zeta'\} \in \mathcal{Z}$$

where π_1, π_2 are the obvious projections. Intuitively, ζ characterises a deadlock ultrafilter whereas $\mathcal{Z}$ characterises all the ultrafilters from which a deadlock state can be reached in finitely many transitions. We then define an **acceptable** set of frame conditions as a set of L_S-formulae which are K_S-consistent and do not contain any deadlocking set of formulae. This definition is extended via the trivial translation to $\mathsf{KKV}(S)$-frame conditions.

Note that ζ characterizes ultrafilters from which no transition is possible at all, even trivial transitions defined by nullary terms in the signature - which also encodes a notion of 'deadlock' - are forbidden.

Theorem 4 (Sahlqvist Completeness for Polynomial Functors). *Let S be a polynomial functor, $\mathcal{L}_S$ be the ∇-language it defines and let $C \subseteq \mathcal{L}_S$ be an acceptable set of Sahlqvist frame conditions, then $\mathsf{KKV}(S) + C$ is complete w.r.t. the class of S-coalgebras validating C.*

Proof (Sketch). We start with a $\mathsf{KKV}(S) + C$-consistent formula a and we will show that we can find a model in the class of S-coalgebras which validate C, in which a is satisfied. Let Σ be the signature defining S.

Finite signatures: The map γ_c defined by Eq. (2) is well-defined because we can derive $\top \leq \bigvee_{\sigma\in\Sigma} \sigma(\top, \ldots, \top)$ in K_S, i.e. ultrafilters always have a successor. Moreover, by the $(\nabla 2)_S$ axioms we cannot have tuples prefixed with different operator symbols in an ultrafilter ϕ since we cannot have $\bot \in \phi$. This guarantees that $\gamma_c(\phi)$ lands in a unique component of the coproduct defining S. It is not too difficult to check that $\gamma_c(\phi)$ is indeed an ultrafilter. The canonical valuation is given as expected by $\pi_c : \mathsf{Uf}(\mathcal{A}_S(C)) \to \mathcal{P}(V), \phi \mapsto V \cap \phi$. Since we have a total function γ_c, rather than a relation like in the traditional Kripke setting, we do not need an Existence Lemma and we can move straight to the Truth Lemma which is easily proven: if we define $\mathcal{M}_c = (\mathsf{Uf}(\mathcal{A}_S(C)), \gamma_c, \pi_c)$, then

$$(\mathcal{M}_c, \phi) \models a \text{ iff } a \in \phi$$

We can now build a model in which a is satisfied: take any ultrafilter ϕ containing a and we have $(\mathcal{M}_c, \phi) \models a$. Now, all we need to do is to show that our canonical model is based on a coalgebra which validates the frame conditions of C. To do this, we need to consider the *complex algebra* associated with $(\mathsf{Uf}(\mathcal{A}(C)), \gamma_c)$. Specifically, we put a Σ-BAO structure on $\mathcal{P}(\mathsf{Uf}(\mathcal{A}_S(C)))$ by defining

$$\sigma^\varepsilon(X_1, \ldots, X_n) = \{\phi \in \mathsf{Uf}(\mathcal{A}_S(C)) \mid \gamma_c(\phi)_i \in X_i, 1 \leq i \leq n\}$$

for all $\sigma \in \Sigma$. The reason for the notation σ^ε, is that the BAO we've just defined is nothing but $\mathcal{A}_S(C)^\varepsilon$, the canonical extension of $\mathcal{A}_S(C)$. Now, since all the formulae in C are Sahlqvist, then they must be canonical (see [6]), i.e. they must all hold in $\mathcal{A}_S(C)^\varepsilon$. It is then easy to check that if a formula of C holds in $\mathcal{A}_S(C)^\varepsilon = \mathcal{P}(\mathsf{Uf}(\mathcal{A}_S(C)))$, it must be valid on the S-frame $(\mathsf{Uf}(\mathcal{A}(C)), \gamma_c)$.

Infinite signatures: To account for the possibility of deadlock states we start by building a slightly different canonical model. The carrier set is given by

$$W_c = \{\phi \in \mathsf{Uf}(\mathcal{A}(C)) \mid \text{ for all } \zeta' \in \mathcal{Z}, \zeta' \not\subseteq \phi\}$$

The map $\gamma_c : W_c \to SW_c$ is then defined as above and is well-defined by construction and by the comments made in the finite signature case. The Truth lemma holds just as in the finite signature case. So to build a model for a we just need to find an ultrafilter $\phi \in W_c$ containing a. It is quite straightforward to check that this is indeed possible (in fact it is possible for any finite set of formulae). The complex algebra associated with W_c is defined as in the finite signature case and is a subalgebra of the canonical extension $(\mathcal{A}(C))^\varepsilon$ of $\mathcal{A}(C)$. Since C is a set of Sahlqvist formulae, they are canonical and thus $(\mathcal{A}(C))^\varepsilon$ belongs to the variety they define. By Birkhoff's theorem this variety is closed under taking subalgebras and so $\mathcal{P}(W_c)$ is an algebra satisfying the equations of C. The fact that (W_c, γ_c) validates the formulae in C follows.

Remark 5. As was hinted in the proof above, if the polynomial functor S is defined by a signature with only finitely many operation symbols, then the result above can be strengthened to a *strong completeness* result, i.e. any consistent set of formulae is satisfiable. In the case of infinite signatures, only finite sets of consistent formulae are guaranteed to be satisfiable. However, *acceptable* sets of formulae in the sense of Definition (3) are also satisfiable, providing a result which is somewhere between weak and strong completeness.

3 Presentations and translations

We will make crucial use of the fact that every accessible functor arises as the quotient of a polynomial functor. By a λ*-ary presentation* of a set-endofunctor T we understand a λ-ary signature (Σ, ar) (i.e. arities are bounded by λ) together with an epi natural transformation $q : S_\Sigma \twoheadrightarrow T$. It is well known that every λ-accessible endofunctor has a λ-ary presentation and we refer the reader to [2] for a detailed overview of presentations in the context of coalgebras.
A natural question to ask in this context is: given a natural transformation $q : S \to T$, what can we say about the relationship between the coalgebraic logics associated with S and T? Is there a syntactic relationship? And what happens at the semantic level? These questions seem natural but, as far as we know, have not really been studied systematically in the literature. Let us first look at what happens at the syntactic level.

Definition 6. Let S, T be two weak-pullback preserving standard functors on **Set** and let $q : S \to T$ be a natural transformation. We define the translation map $(\cdot)^q : \mathcal{L}_S \to \mathcal{L}_T$ recursively by

$$(\nabla\alpha)^q = \nabla(q_{\mathcal{L}_T} \circ S(\cdot)^q)(\alpha)$$

We call $(\cdot)^q$ the **translation along** q and will use the following notational conventions for maps associated with $(\cdot)^q : \mathcal{L}_S \to \mathcal{L}_T$

- $\langle.\rangle^q : S\mathcal{L}_S \to T\mathcal{L}_T$ will be shorthand for the map $q_{\mathcal{L}_T} \circ S(\cdot)^q$
- $[\cdot]^q : \mathcal{P}_\omega S\mathcal{L}_S \to \mathcal{P}_\omega T\mathcal{L}_T$ will be shorthand for the map $\mathcal{P}_\omega\langle.\rangle^q$
- $\{\cdot\}^q : S\mathcal{P}_\omega\mathcal{L}_S \to T\mathcal{P}_\omega\mathcal{L}_T$ will be shorthand for the map $q_{\mathcal{P}_\omega\mathcal{L}_T} \circ S\mathcal{P}_\omega(\cdot)^q$

Note that with this notation we have $(\nabla\alpha)^q = \nabla(\langle\alpha\rangle^q)$.

At the level of the semantics, note that a natural transformation $q : S \to T$ induces a functor $Q : \mathbf{CoAlg}(S) \to \mathbf{CoAlg}(T)$ on the corresponding categories of coaglebras, given by $Q(W, \gamma) = (W, q_W \circ \gamma)$. In particular Q turns models for $\mathcal{L}_S$-formulae into models for $\mathcal{L}_T$-formulae and we will now show that the translation along q agrees with the functor Q in the sense that truth is preserved by applying both simultaneously. Formally:

Proposition 7. *Suppose that $q : S \to T$ is a natural transformation and that $a \in \mathcal{L}_S$. Suppose also that we have a model $\mathcal{M} = (W, \gamma, \pi)$ such that*

$$\mathcal{M}, w \models a$$

for some $w \in W$. If we then define $Q(\mathcal{M}) = (W, q_W \circ \gamma, \pi)$ we have

$$Q(\mathcal{M}), w \models (a)^q$$

The following lemma shows how the functors Base^S and Base^T are related by an epi natural transformation $q : S \twoheadrightarrow T$. This lemma will be very useful to relate concepts for S and T which depend on the bases.

Proposition 8. *Let S, T be* **Set** *functors and $q : S \twoheadrightarrow T$ an epi natural transformation between them, let T weakly preserve pullbacks and let $\mathcal{L}_S$ and $\mathcal{L}_T$ be the ∇-languages induced by S and T respectively, then the following diagram commutes:*

$$\begin{tikzcd} S\mathcal{L}_S \arrow[r, "\mathsf{Base}^S"] \arrow[d, "\langle\cdot\rangle^q"'] & \mathcal{P}\mathcal{L}_S \arrow[d, "\mathcal{P}(\cdot)^q"] \\ T\mathcal{L}_T \arrow[r, "\mathsf{Base}^T"] & \mathcal{P}\mathcal{L}_T \end{tikzcd}$$

We conclude this section with an example.

Example 9. A functor of particular interest for applications is the so-called bag functor which we denote $\mathcal{B}$. Coalgebras for the bag functor are models for **Graded Modal Logic** which is essentially the modal logic version of *cardinality restrictions* in Description Logics. The bag functor can be defined by

$$\mathcal{B}X = \{f : X \to \mathbb{N} \mid supp(f) \text{ is finite }\}$$

Alternatively and equivalently, an element of $\mathcal{B}X$ can be defined as a 'multiset', i.e. a set of pairs (denoted with ':') $\{(x_i : n_i) \mid i \in I\}$ where the elements x_i , $i \in I$ are distinct elements of X and the $n_i, i \in I$ are integers thought of as the multiplicities of the elements x_i. $\mathcal{B}$ has a simple presentation in terms of the list functor $\mathsf{List}X = \coprod_{n\in\omega} X^n$. The presentation is given by:

$$q_X : \mathsf{List}X \to \mathcal{B}X, (a_1, \ldots, a_n) \mapsto \{(a_{p(1)}, \ldots, a_{p(n)}) \mid p \in \mathsf{Perm}(n)\}$$

where $\mathsf{Perm}(n)$ is the group of permutations of n elements. In other words q identifies all permutations of a given tuple, and thus an element of $\mathcal{B}X$ can be represented as a multiset $(a_1 : k_1, \ldots, a_n : k_n)$ where $a_i : k_i$ means a_i appears k_i times in the (equivalence class of) tuple. In this context the translation $(\cdot)^q$ works as follows: a $\mathcal{L}_{\mathsf{List}}$-formula of the form $\nabla n(a_1, \ldots, a_n)$ gets translated into an $\mathcal{L}_{\mathcal{B}}$-formula of the shape $\nabla n(k_1 : (a'_1)^q, \ldots, k_m : (a'_m)^q)$ where the a'_i are the distinct elements of the set $\{a_1, \ldots, a_n\}$ and k_i their multiplicity (and $m \leq n$).

4 The Translation Theorem

We have so far established the following: (1) we have a logic for coalgebras based on polynomial functors which is suitable for defining Sahlqvist formulae and (2) every accessible functor T can be presented from a polynomial functor S and this presentation allows us to move from the language and the coalgebras based on S to those based on T in a sensible way. This section is they key technical contribution of this paper and shows how we can connect the facts that we have established in the two previous sections. The idea will be to show that the translation map $(\cdot)^q$ acts not just on the *language* $\mathcal{L}_S$ but also on the *logic* $\mathsf{KKV}(S)$. Crucially, we will show how derivability in $\mathsf{KKV}(S)$ is related to derivabilty in $\mathsf{KKV}(T)$. This section is rather technical and we must start with a few lemmata. The intended meaning of our first lemma is that $(\cdot)^q$ sends substitution instances of axioms to substitution instances of axioms.

Definition 10. A substitution is a map $\hat{\pi} : \mathcal{L}_T \to \mathcal{L}_T$ defined inductively from a map $\pi : V \to \mathcal{L}_T$ by: $\hat{\pi}(p) = \pi(p)$ for all $p \in V$, $\hat{\pi}(\phi \wedge \psi) = \hat{\pi}(\phi) \wedge \hat{\pi}(\psi)$, $\hat{\pi}(\neg\phi) = \neg\hat{\pi}(\phi)$ and $\hat{\pi}(\nabla\alpha) = \nabla(T\hat{\pi}(\alpha))$.

Lemma 11. *Let S, T be two weak-pullback preserving functors on* **Set***, let $q : S \twoheadrightarrow T$ be a epi natural transformation and let $\mathcal{L}_S$ and $\mathcal{L}_T$ be the ∇-languages induced by S and T respectively. Let $\pi : V \to \mathcal{L}_S$ define a substitution $\hat{\pi} : \mathcal{L}_S \to \mathcal{L}_S$ and and let ρ be the map $\rho = (\cdot)^q \circ \pi : V \to \mathcal{L}_T$, then for all $a \in \mathcal{L}_S$*

$$(\cdot)^q \circ \hat{\pi}(a) = \hat{\rho} \circ (\cdot)^q(a)$$

There are two important constructions in the KKV axiomatization: the notion of slim redistribution and that of lifted member (used for the ($\nabla 2$) and ($\nabla 3$) axiom). The following two lemmata show how these notions interact with the translation map. They are generalisation of Lemmata 5.44 and 5.45 in [10] where a special class of presentations (called 'well-based' presentations) is considered, here we consider arbitrary epi natural transformations between weak-pullback preserving functors.

Lemma 12. *Let S, T be two weak-pullback preserving functors on* **Set***, let $q : S \twoheadrightarrow T$ be an epi natural transformation, and let $\mathcal{L}_S$ and $\mathcal{L}_T$ be the ∇-languages induced by S and T respectively. For any $A \in \mathcal{P}_\omega S\mathcal{L}_S$ and $\Phi \in T\mathcal{P}_\omega\mathcal{L}_T$, the following two conditions are equivalent:*

(1) $\Phi \in SRD([A]^q)$
(2) there exist $\Phi' \in S\mathcal{P}_\omega\mathcal{L}_S$ such that $\{\Phi'\}^q = \Phi$ and for all $\alpha \in A$ there exist α' such that $\alpha' \,\bar{S}\!\in \Phi'$ and $\langle\alpha'\rangle^q = \langle\alpha\rangle^q$

Lemma 13. *Let S, T be two weak-pullback preserving functors on* **Set***, let $q : S \twoheadrightarrow T$ be an epi natural transformation, and let $\mathcal{L}_S$ and $\mathcal{L}_T$ be the ∇-languages induced by S and T respectively. For any $\alpha \in T\mathcal{L}_T$ and $\Phi \in T\mathcal{P}_\omega\mathcal{L}_T$ the following two conditions are equivalent:*

(1) $\alpha \,\bar{T}\!\in \Phi$ for $\Phi \in T\mathcal{P}_\omega\mathcal{L}_T$
(2) there exist $\Phi' \in T\mathcal{P}_\omega\mathcal{L}_S$ such that $\{\Phi'\}^q = \Phi$ and $\alpha' \in S\mathcal{L}_S$ such that $\langle\alpha'\rangle^q = \alpha$ and $\alpha' \,\bar{S}\!\in \Phi'$

We are now ready to move to our key technical result. Our main motivation is to get a completeness result for $\mathsf{KKV}(T) + C$ where C is a set of 'Sahlqvist formulae' - we will define what this means precisely in the next section. Following the usual method we'll start with a $\mathsf{KKV}(T) + C$-consistent formula a and try to build a model for it. By Theorem 4, we know how to do this for $\mathsf{KKV}(S) + D$, when $S \twoheadrightarrow T$ is a presentation of T and D is a set of Sahlqvist formulae. So what seems to be required is a result linking $\mathsf{KKV}(T) + C$-consistency to $\mathsf{KKV}(S) + D$-consistency for the right D. More specifically, we want a result relating $\neg a \leq \bot$ not being derivable in $\mathsf{KKV}(T) + C$ to a similar statement in $\mathsf{KKV}(T) + D$ for a certain D. As it turns out, the trick is to look at *all* the pre-images of a and of C, and, using the contrapositive, the result we are looking for is therefore:

Theorem 14 (Translation Theorem). *Let T be a weak-pullback preserving* **Set** *functor, let $q : S \twoheadrightarrow T$ be a presentation of T and let $\mathcal{L}_S$ and $\mathcal{L}_T$ be the ∇-languages defined by S and T. Assume we have a set C of $\mathsf{KKV}(T)$-consistent formulae (the frame conditions) and let us define the set $C' \subseteq \mathcal{L}_S$ by:*

$$C' = \{c' \in \mathcal{L}_S \mid (c')^q \in C \text{ and } c' \text{ is } \mathsf{KKV}(S) - \textit{consistent}\}$$

Then

$$\mathsf{KKV}(S) + C' \vdash \{a' \leq \bot \mid (a')^q = a\}$$

implies

$$\mathsf{KKV}(T) + C \vdash a \leq \bot$$

Proof (Sketch). We proceed by induction on the depth n of the shortest $\mathsf{KKV}(S)+C'$-proof of $a' \leq \bot$ amongst all a' such that $(a')^q = a$. The base case is if $n = 0$, i.e. if there there exist an inequality $a' \leq \bot$ which is either an axiom of the propositional fragment of the logic or a substitution instance of an axiom in C'. By Lemma 11 and the definition of C', it is clear that if a' is a substitution instance of a formula $c' \in C'$, then its translation $(a')^q = a$ is a substitution instance of a formula in $c \in C$, and we can thus conclude that $\mathsf{KKV}(T) + C \vdash a \leq \bot$. The inductive hypothesis is the following: if we have $\mathsf{KKV}(S) + C'$ proofs that all the pre-images under $(\cdot)^q$ of a formula a are false and if the smallest of these proofs has depth n, then we have a proof that a is false in $\mathsf{KKV}(T) + C$. So let's assume that we have proofs

$$\mathsf{KKV}(S) + C' \vdash \{a' \leq \bot \mid (a')^q = a\}$$

and that the depth of the shortest proof if $n + 1$. We then show that we can always find a set of $\mathsf{KKV}(S) + C'$-proofs of minimal depth n whose conclusion are the pre-images of a premise in $\mathsf{KKV}(T) + C$ whose conclusion is $a \leq \bot$. In other words, we build the last step of a T-proof by using the last steps of existing S-proofs and the inductive hypothesis. This is done by examining, in turn, each of the possible outermost connectives of a, and thus of a', i.e. ∇ or a boolean connective. Each of these possible outermost connectives of a' specifies a small number of rules which could have been the last rule applied to reach $(a' \leq \bot)$. The proof then consists in examining each of these possibilities and show that they all lead to a situation where the induction hypothesis can be applied and lead to a T-rule with conclusion $a \leq \bot$.

5 Sahlqvist formulae for Coalgebraic Logics

We now have all we need to formulate our Sahlqvist completeness result for coalgebraic modal logic. We start by defining a notion of Sahlqvist formula for a general (i.e. not necessarily polynomial) functor.

Definition 15. Let T be a weak-pullback preserving functor, let $q : S \twoheadrightarrow T$ be a presentation of T and let $\mathcal{L}_T$ and $\mathcal{L}_S$ be the ∇-languages induced by S and T respectively, then $a \in \mathcal{L}_T$ will be called a **coalgebraic Sahlqvist formula** if every pre-image of a under the translation map $(\cdot)^q : \mathcal{L}_S \to \mathcal{L}_T$ is Sahlqvist in the sense of Definition 2. A set $C \subseteq \mathcal{L}_T$ will be called an **acceptable** set of frame conditions if its inverse image under $(\cdot)^q$ is acceptable in the sense of Definition 3.

Theorem 16 (Sahlqvist Completeness Theorem). *Let T be a weak-pullback preserving* **Set** *functor, let $q : S \twoheadrightarrow T$ be a presentation of T and let $\mathcal{L}_T$ and $\mathcal{L}_S$ be the ∇-language induced by S and T respectively. Assume that $C \subseteq \mathcal{L}_T$ is an acceptable set of coalgebraic Sahlqvist formulae, then $\mathsf{KKV}(T) + C$ is complete w.r.t. the class of T-coalgebra validating C.*

Proof. As is customary, we start with a formula $a \in \mathcal{L}_T$ which is $\mathsf{KKV}(T) + C$-consistent, and we will build a model for a in the class of T-coalgebras validating the coalgebraic frame conditions in C. The proof is in four steps.
Firstly, by using the contrapositive of Theorem 14, we know that since a is $\mathsf{KKV}(T) + C$-consistent, then there must exist a pre-image a' of a under $(\cdot)^q$ which is $\mathsf{KKV}(S) + C'$-consistent.
Secondly, since C is a set of coalgebraic Sahlqvist formulae we can apply Theorem 4 and conclude that there exists a model $\mathcal{M}_S$ based on an S-coalgebra (W, γ) which belongs to the class of coalgebraic frames validating the axioms of C' and such that a' is satisfied in $\mathcal{M}_S$, i.e. there exist $w \in W$ such that $\mathcal{M}_S, w \models a'$.
Thirdly, by Proposition 7 if we define $\mathcal{M}_T = Q(\mathcal{M}_S)$ then we have that since $\mathcal{M}_S, w \models a'$ and $(a')^q = a$, $\mathcal{M}_T, w \models a$.
Finally, we need to check that $\mathcal{M}_T$ is a coalgebraic frame validating the formulae in C. Assume that it is not, then there must exist $c \in C$ and $w \in W$ such that $\mathcal{M}_T, w \not\models c$. By the contrapositive of Proposition 7 this means that $\mathcal{M}_S, w \not\models c'$ for all c' in the pre-image of c under $(\cdot)^q$. But by Definition 15 and the second step of the proof we know that we must have $\mathcal{M}_S, w \models c'$ for any such c' and we therefore have a contradiction.

Example 17. We return to the bag functor $\mathcal{B}$ of Example 9 to illustrate what is, as far as we know, the first Sahlqvist completeness result for Graded Modal Logic (GML). An $\mathcal{L}_\mathcal{B}$-formula of the shape $\nabla n(a_1 : k_1, \ldots, a_m : k_m)$ is true at a point w if w has n successors, of which k_i satisfy a_i, for $1 \leq i \leq m$. Note that by construction ($\mathcal{B}$ is presented by List) our $\mathcal{B}$-coalgebras are finitely branching. For this example we will place ourselves in the predicate lifting style logic obtained by the trivial translation removing ∇ operators (see Example 1), and we rewrite $\nabla n(a_1 : k_1, \ldots, a_m : k_m)$ as $\langle n \rangle(a_1 : k_1, \ldots, a_m : k_m)$. For clarity's sake we define the following derived modal operators which are closer to the traditional operators of GML:

$$\Diamond_n p = \bigvee_{i=1}^{n} \langle n \rangle (p : i, \top : (n-i))$$
$$\Diamond_{\leq n} p = \bigvee_{i=1}^{n} \Diamond_n p$$

Thus $\Diamond_n p$ holds at w if w has n successors and p is true at (at least) one of them, whereas $\Diamond_{\leq n} p$ holds at w if w has at most n successors and p is true at (at least) one of them, i.e. p is true at at least one and at most n successors. Using these operators we can define graded versions of the most popular Sahlqvist frame conditions, for example 'transitivity for at most n successors':

$$(4_n): \quad \Diamond_{\leq n} \Diamond_{\leq n} p \to \Diamond_{\leq n} p$$

To see that (4_n) is Sahlqvist, note first that all the pre-images of $\langle n \rangle (p : i, \top : (n-i))$ under the translation map $(\cdot)^q$ introduced in Example 9 are of the shape (in the predicate-lifting style):

$$n(\pi(\underbrace{p, \dots, p}_{i \text{ times}}, \underbrace{\top, \dots, \top}_{(n-i) \text{ times}})) \tag{3}$$

for some $\pi \in \mathsf{Perm}(n)$. i.e. just an operator applied to some variables. So a pre-image of $\Diamond_n p$ is just a join of n formulae of the shape (3) for a choice of n permutations $\pi_i \in \mathsf{Perm}(n), 1 \leq i \leq n$ (or combinations of elements of this shape using meets and joins). In turn, the pre-images of $\Diamond_{\leq n} p$ are joins of n choices of pre-images of $\Diamond_i p, 1 \leq i \leq n$ (or combinations of elements of this shape using meets and joins). Thus the pre-images of the consequent of (4_n) are (strictly) positive. Similarly, the antecedent of (4_n) can be seen to be strictly positive and thus (4_n) is a Sahlqvist formula in the sense of Eq. (1) for any n.
Note that the cardinality restriction leads to a slightly counter-intuitive meaning for the axiom (4_n). Indeed, assume a point w has two successors, that one of these successors has three successors, one of which is the only state to satisfy p, then (4_2) holds, but transitivity doesn't. So (4_n) is transitivity for frames with branching degree at most n. To recover the usual notion of transitivity we need to consider the collection of Sahlqvist formulae $(4) = \{(4_n) \mid n \in \mathbb{N}\}$. It is clear that (4) is acceptable in the sense of Definition 3 and the basic GML + (4) is thus weakly complete w.r.t. finitely branching transitive frames.

Remark 18. We must make two important remarks about the previous example. First, the fact that $\mathcal{L}_{\mathcal{B}}$-formulas count the total number of successors points to an important difference with the traditional language for Graded Modal Logic $\mathcal{L}_{\mathsf{GML}}$ where a formula $\Diamond_k \phi$ is traditionally interpreted as 'ϕ holds at k distinct successors', leaving the total number of successors unspecified. Clearly we cannot express this in a finitary way in $\mathcal{L}_{\mathcal{B}}$, so our Sahlqvist formulae are expressed in a fragment of $\mathcal{L}_{\mathsf{GML}}$. But there is a translation tr from $\mathcal{L}_{\mathcal{B}}$ to $\mathcal{L}_{\mathsf{GML}}$ defined by

$$\mathsf{tr}(\nabla n(a_1 : k_1, \ldots, a_m : k_m)) = \Diamond_n \top \wedge \Box_{n+1} \bot \wedge \bigwedge_{i=1}^{m} \Diamond_{k_i} a_i$$

where $\Diamond_n \top \wedge \Box_{n+1} \bot$ just says that there are exactly n successors. Our second remark is that the $\Diamond_k$ modalities are algebraically ill-behaved as they do not distribute over joins, so there is no way of applying the theory of Sahlqvist formulae in BAOs to GML in the usual setting which may explain why we were unable to find any Sahlqvist completeness result for this logic in the literature.

Example 19. Our next example, is intended to show the relationship between our notion of Sahlqvist formula and the traditional one from relational modal logic. Here we will look at the finite powerset functor $\mathcal{P}_\omega$ which has a very simple presentation $q : \mathsf{List} \twoheadrightarrow \mathcal{P}_\omega$ given by

$$q_X : \mathsf{List}X \twoheadrightarrow \mathcal{P}_\omega X, (a_1, \ldots, a_n) \mapsto \{a_1, \ldots, a_n\}$$

The empty list is sent to the empty set. Thus, the pre-images of a $\mathcal{L}_{\mathcal{P}_\omega}$-formula of the type $\nabla\{a_1, \ldots, a_k\}$ are all the $\mathcal{L}_{\mathsf{List}}$-formula of the type $\nabla n(a'_1, \ldots, a'_n), n \geq k$ where $(a'_1, \ldots, a'_n)$ is any list containing all the elements of $\{a_1, \ldots, a_k\}$. Here we are in a slightly better position than in the graded case as there are semantic-preserving translations of the usual modal language $\mathcal{L}_{ML}$ in terms of $\Diamond$ and $\Box$ into $\mathcal{L}_{\mathcal{P}_\omega}$ and vice-versa (see [9]), in particular $\Diamond p$ is translated by $\nabla\{p, \top\}$ and $\Box p$ by $\nabla\emptyset \vee \nabla\{p\}$. We can check that the traditional Sahlqvist formulae as defined for example in [4] are also Sahlqvist formulae in the sense of this paper. Notice first that the pre-images under $(\cdot)^q$ of $\Diamond p$, or equivalently of $\nabla\{p, \top\}$, are of the shape $\nabla\alpha$ for an $\alpha \in \mathsf{List}\{p, \top\}$, or, in the predicate-lifting style, $\langle n \rangle \alpha$ with $\alpha \in (\{p, \top\})^n$ for some $n \in \mathbb{N}$. It is then quite straightforward to check that positive formulae in $\mathcal{L}_{ML}$ are translated into positive formulae in $\mathcal{L}_{\mathcal{P}_\omega}$ whose inverse images under $(\cdot)^q$ are also positive. This takes care of the consequent of Sahlqvist formulae. Now for the antecedent. As defined in [4], the antecedent must be built from $\bot, \top$, boxed atoms and negative formulae using $\wedge, \vee$ and $\Diamond$. Clearly, $\bot$ and $\top$ pose no problem. Negative $\mathcal{L}_{ML}$-formulae get mapped to negative $\mathcal{L}_{\mathcal{P}_\omega}$-formulae whose inverse image under $(\cdot)^q$ are also negative. The only potentially problematic building block are the boxed atoms. The formula $\Box p$ is translated to $\nabla\emptyset \vee \nabla\{p\}$ whose inverse images under $(\cdot)^q$ are of the shape (in the predicate lifting style) $\langle 0 \rangle \vee \langle n \rangle (p, \ldots, p)$, i.e. *strictly* positive terms. Clearly, the nesting of more boxes doesn't change this and so all inverse images of boxed atoms are strictly positive and we can therefore view them as part of the strictly positive term in Eq. (1) defining Sahlqvist antecedents.

6 Outlook

As illustrated by Example 17, there are instances of logics in the predicate-lifting style which can make statements that cannot be translated in the ∇-style (see

[10]). We would like to extend our result to such logics, possibly by enriching the polynomial logics with operators that carry an infinitary meaning but remain algebraically well-behaved. We would also like to extend our result to richer coalgebraic logics such as coalgebraic μ-calculus (see [3] for recent advances in defining Sahlqvist formulae for the μ-calculus) and hybrid coalgebraic modal logic. Finally we would like to find examples and applications of our results to more logics such as probabilistic or coalition logics.

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